Exact Solutions for a Spiral Wave in the Six-Dimensional Phase Space Model

Abstract

Spiral (vortex) waves have recently found growing practical applications: in biology and medicine — as the mechanism of cardiac rhythms (spiral waves of excitation in cardiac tissue), in data transmission — as carriers with orbital angular momentum in optical and acoustic communication, and in astronomical observations — in vortex coronagraphs for suppression of stellar light. This motivates the problem of constructing exact solutions for waves of this type.

In the present work, for a linear matrix wave equation with a pair of operators W (symmetric) and S (antisymmetric), an exact solution is constructed in the form of a matrix exponential. It is proven that the solution satisfies simultaneously the Helmholtz equation and the Klein–Gordon-type equation; that it is unitary; that the full system is neutrally stable; that the solution space is finite-dimensional (12 complex modes); that the tachyonic branch of the spectrum is suppressed by the constraint; that in light-cone coordinates the system decomposes into three equations

reducible entirely to the algebra of the matrices; and that quantization is carried out exactly, yielding a positive Hamiltonian. All statements are proven by direct computation, without approximations or assumptions.

1. Statement of the Problem

The matrices are given by

$$W = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Properties:

$$W^T = W, \quad S^T = -S,$$

$$\sigma(W) = \{2, -1, -1\}, \quad \sigma(S) = \{0, -i\sqrt{3}, +i\sqrt{3}\}.$$

The key identity (verified by direct multiplication):

$$WS = SW = -S.$$

Structure of the matrices W and S

The matrices W and S are not chosen arbitrarily — they are assembled from three elementary matrices. Define the real symmetric 3×3 matrices whose nonzero entries equal unity:

$$(M_1)_{11} = (M_1)_{23} = (M_1)_{32} = 1,$$

$$(M_2)_{13} = (M_2)_{22} = (M_2)_{31} = 1,$$

$$(M_3)_{12} = (M_3)_{21} = (M_3)_{33} = 1.$$

Explicitly:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Each of them is real-symmetric and hence Hermitian, orthogonal and hence unitary, and each is its own inverse:

$$M_i^T = M_i, \qquad M_i^2 = I, \qquad M_i^{-1} = M_i.$$

These are the three permutation matrices corresponding to the transpositions of the group S_3.

Now the main point. For any pair of these three matrices, the anticommutator yields one and the same matrix W, and the commutator yields one and the same matrix S:

$$\{M_1, M_2\} = \{M_2, M_3\} = \{M_3, M_1\} = W,$$

$$[M_1, M_2] = [M_2, M_3] = [M_3, M_1] = S,$$

where

$$\{A, B\} = AB + BA, \qquad [A, B] = AB - BA.$$

This is verified by direct multiplication; for instance,

$$M_1 M_2 = \tfrac{1}{2}(W + S), \qquad M_2 M_1 = \tfrac{1}{2}(W - S).$$

Note in passing: the intermediate products $M_1 M_2$ and $M_2 M_1$ are the two remaining permutation matrices (the three-cycles). Together with the identity matrix and the three M_i, they exhaust all six permutations of the group S_3. Thus the pair (W, S) is not arbitrary but arises naturally from the algebra of permutations: the symmetric part of the product of transpositions is W, and the antisymmetric part is S.

We seek a solution $U(x, y)$ — a unitary 3×3 matrix depending on two scalar arguments.

2. The Solution Operator

$$U(x,y) = e^{A(x,y)}, \quad A = iW(x+y) + S(y-x).$$

Derivatives of the operator:

$$\frac{\partial A}{\partial x} = iW - S, \quad \frac{\partial A}{\partial y} = iW + S,$$

$$\left[\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}\right] = 0.$$

Commutativity follows from $WS = SW$ and yields the exact differentiation formula for the exponential:

$$\frac{\partial U}{\partial x} = \left(\frac{\partial A}{\partial x}\right) U,$$

$$\frac{\partial U}{\partial y} = \left(\frac{\partial A}{\partial y}\right) U.$$

Unitarity is an exact consequence of the anti-Hermiticity of A ($A^\dagger = -A$):

$$U^\dagger U = I.$$

3. Derivatives of the Solution

First derivatives:

$$U'_x = (iW - S)\,U, \quad U'_y = (iW + S)\,U,$$

$$U'_x + U'_y = 2iW\,U, \quad U'_y - U'_x = 2S\,U.$$

Auxiliary identities (verified by direct multiplication):

$$W^2 = J + I, \quad S^2 = J - 3I,$$

$$W^2 + S^2 = 2W, \quad W^2 - S^2 = 4I,$$

where J is the all-ones matrix. The first identity shows that $W + 2I = W^2$ and $W - 2I = S^2$, to be used below.

Second derivatives:

$$U''_{xx} = (-4 + 2iS)\,U,$$

$$U''_{yy} = (-4 - 2iS)\,U,$$

$$U''_{xy} = U''_{yx} = -2W\,U.$$

4. The Fundamental Equations

Adding and subtracting the second derivatives, we obtain two closed systems:

$$U''_{xx} + U''_{yy} = -8\,U,$$

$$U''_{xx} - U''_{yy} = 4iS\,U.$$

The first is the matrix Helmholtz equation, the second is a Klein–Gordon-type equation with a matrix "mass." In addition:

$$U''_{xy} = -2W\,U.$$

All three equations hold simultaneously — the system is overdetermined but consistent.

Light-cone coordinates: the system closes

This overdetermined system has a consequence that makes the picture complete — and it is revealed through the mixed derivative. Take the Helmholtz equation and the mixed equation and form their combination with coefficient -2 on U''_{xy}:

$$U''_{xx} + U''_{yy} - 2U''_{xy} = -8\,U + 4W\,U = 4(W - 2I)\,U.$$

By the identity noted above, $W - 2I = S^2$. Hence:

$$U''_{xx} + U''_{yy} - 2U''_{xy} = 4S^2U.$$

Now pass to light-cone coordinates:

$$s = x + y, \quad t = x - y.$$

In these coordinates the derivatives rearrange as $\partial_x = \partial_s + \partial_t$, $\partial_y = \partial_s - \partial_t$, and the left-hand sides of our combinations turn into

ordinary second derivatives along s and t:

$$\partial_{xx} + \partial_{yy} - 2\partial_{xy} = 4\,\partial^2_{tt},$$

$$\partial_{xx} + \partial_{yy} + 2\partial_{xy} = 4\,\partial^2_{ss}.$$

The first combination gives the equation along the light-cone direction t:

$$U''_{tt} = S^2\,U.$$

The second — along s:

$$U''_{ss} = -W^2\,U,$$

(here we used $W + 2I = W^2$). Finally, from the difference of the Klein–Gordon and Helmholtz equations one obtains the mixed relation:

$$U''_{st} = iS\,U.$$

Thus, the entire system in light-cone coordinates consists of three equations:

$$U''_{ss} = -W^2\,U,$$

$$U''_{tt} = +S^2\,U,$$

$$U''_{st} = iS\,U.$$

The mass spectra along each direction are read directly from the squared matrices:

$$\sqrt{\sigma(W^2)} = \{2,\, 1,\, 1\},$$

$$\sqrt{-\sigma(S^2)} = \{0,\, \sqrt{3},\, \sqrt{3}\},$$

the operator $-S^2$ is positive semidefinite, so the dynamics along t is purely oscillatory. The identity $W^2 - S^2 = 4I$ glues the two directions back into Helmholtz:

$$U''_{ss} + U''_{tt} = (-W^2 + S^2)\, U = -4\, U,$$

and for each mode this means the on-shell relation: the squared frequency along s minus the squared frequency along t equals 4.

The physical meaning is easiest to see along the ray $x + y = \mathrm{const}$, that is, at $ds = 0$: there the field obeys $U''_{tt} = S^2 U$, and its solution splits by projectors into two very different parts. The neutral P_0-component satisfies $u'' = 0$ — it simply stays in place. The charged $P_\pm$-components satisfy $u'' = -3u$ — they rotate uniformly with frequency $\sqrt{3}$. The picture is remarkable: the field travels at the

speed of light while "spinning inside itself" — the neutral part is at rest, the charged part rotates. The effective mass along t arises not from an external potential but from this internal rotation — a familiar theme of gauge theories, here visible with the naked eye.

It is important to understand the status of what has been obtained: the equation $U''_{tt} = S^2 U$ is not a new condition but a consequence of the original ones. The solution space has not narrowed — the same 12 complex modes. However, the system has acquired a closed formulation: in light-cone coordinates the

entire dynamics reduces to the algebra of the matrices W and S, with no residue. The solution is written as a product of two exponentials along the independent directions:

$$U = e^{iWs}\, e^{-St},$$

and the three equations above are simply three projections of this fact: each follows by differentiating with respect to s, with respect to t, and with respect to both at once.

5. Explicit Solution via Projectors

Set $s = x + y, t = x - y$. The projectors onto the eigenspaces of S:

$$P_0 = \tfrac{1}{3} J,$$

$$P_+ = \tfrac{1}{2}\left(I - \tfrac{1}{3}J - \tfrac{iS}{\sqrt{3}}\right),$$

$$P_- = \tfrac{1}{2}\left(I - \tfrac{1}{3}J + \tfrac{iS}{\sqrt{3}}\right),$$

$$P_0 + P_+ + P_- = I, \quad P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha.$$

The correctness of the projectors rests on the identity

$$\left(\frac{iS}{\sqrt{3}}\right)^2 = I - \tfrac{1}{3}J.$$

The explicit solution:

$$U = e^{2is}P_0 + e^{-is}\left(e^{-i\sqrt{3}t}\,P_+ + e^{+i\sqrt{3}t}\,P_-\right).$$

6. Mode Spectrum and Structure of the Solution

The solution is a superposition of three plane waves with wave vectors:

$$k_0 = (2,\ 2),$$

$$k_+ = (-1-\sqrt{3},\ \sqrt{3}-1),$$

$$k_- = (\sqrt{3}-1,\ -1-\sqrt{3}).$$

Checks:

$$k_0 + k_+ + k_- = (0,0),$$

$$|k_0|^2 = |k_+|^2 = |k_-|^2 = 8.$$

All three modes lie on a single circle $k_x^2 + k_y^2 = 8$ — this is the on-shell content of the Helmholtz constraint.

With x chosen as time, the squared masses $m^2 = k_x^2 - k_y^2$ are

$$m_0^2 = 0, \quad m_+^2 = +4\sqrt{3}, \quad m_-^2 = -4\sqrt{3},$$

and the values coincide with the eigenvalues of the operator $4iS$ on the corresponding branches — the Klein–Gordon constraint holds mode by mode.

7. Dimension of the Solution Space

The ansatz $U = e^{\alpha x + \beta y} v$ with an eigenvector v of the matrix S gives, for each branch,

$$\alpha^2 = -(4 \mp 2\sqrt{3}), \quad \beta^2 = -(4 \pm 2\sqrt{3}),$$

that is, 4 exponential modes per branch. In total:

$$3 \times 4 = 12 \text{ complex modes},$$

$$24 \text{ real degrees of freedom}.$$

The eigenvectors are the Fourier vectors of the cyclic group:

$$v_0 = \tfrac{1}{\sqrt{3}}(1,1,1),$$

$$v_+ = \tfrac{1}{\sqrt{3}}(1,\omega^2,\omega),$$

$$v_- = \tfrac{1}{\sqrt{3}}(1,\omega,\omega^2),$$

$$\omega = e^{2\pi i/3}.$$

The constructed solution e^A is a two-dimensional submanifold (a group) inside the 24-dimensional linear space of solutions, singled out by the unitarity condition.

8. Stability

The second-derivative operators $-4 \pm 2iS$ are Hermitian, with spectra

$$\sigma(-4+2iS) = \sigma(-4-2iS) = \{-4,\ -4-2\sqrt{3},\ -4+2\sqrt{3}\},$$

$$-4+2\sqrt{3} = -(\sqrt{3}-1)^2 < 0.$$

All three eigenvalues are negative — the equation $U'' = AU$ with $A \prec 0$ has purely oscillatory dynamics. The frequencies:

$$\omega_0 = 2, \quad \omega_+ = \sqrt{3}+1, \quad \omega_- = \sqrt{3}-1,$$

$$(\sqrt{3}-1)^2 = 4-2\sqrt{3}, \quad (\sqrt{3}+1)^2 = 4+2\sqrt{3}.$$

Conclusion: the full system is neutrally stable, all solutions are bounded in modulus; the stability margin is

$$4 - 2\sqrt{3} \approx 0.54 > 0.$$

The tachyonic band arises only in the KG subsystem in isolation (for $|k| < 2 \cdot 3^{1/4}$) and is completely removed by the second constraint: in the full system the momentum spectrum is discrete, and no instability band exists.

9. Causality

The admissible momenta form a finite set:

$$|k| \in \{2,\ \sqrt{3}-1,\ \sqrt{3}+1\},$$

with two directions per branch. The formal group velocity of the tachyonic branch, $v_g = 2+\sqrt{3} > 1$, produces no superluminal signal: two discrete points cannot be assembled into a wave packet, and the signal has no front. The neutral branch P_0 lies

exactly on the light cone $m^2 = 0$ and defines the causal structure of the model.

Geometric consequence: along the lines $x + y = \mathrm{const}$ the solution takes the form

$$U(\lambda) = e^{A_0}\, e^{-S\, t(\lambda)},$$

that is, a stationary P_0-component (the axis) plus uniform rotation of the $P_\pm$-component in the plane perpendicular to $(1, 1, 1)$, with period

$$T = \frac{\pi}{\sqrt{3}}.$$

The trajectory of the field in $\mathbb{R}^6 \cong \mathbb{C}^3$ is an exact circular helix.

A traveling wave, not a standing one

The picture obtained requires one clarification that removes a natural misunderstanding. It might seem that the stationarity of the P_0-component along the light ray means the wave has turned into a standing one. This is not so: the solution remains a traveling

wave, and the "stationarity" is a property of the method of observation, not of the wave itself.

First, the neutral component e^{2is} is an ordinary traveling plane wave. With x chosen as time, its phase velocity is

$$v_0 = \frac{k_y}{\omega} = \frac{2}{2} = 1,$$

that is, exactly the speed of light — as befits a massless mode on the cone. It appears stationary along the ray $s = \mathrm{const}$ for the simple reason that this ray is its own trajectory: any observer

moving together with the wave sees a constant phase. Step off the ray — and the phase is set in motion at once.

Second, the charged components are also traveling plane waves with fixed wave vectors from Section 6. For instance, the phase velocity of the branch P_+:

$$v_+ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} \approx 0.27 < 1,$$

— subluminal, as for an ordinary massive particle. Along the light ray these components do not propagate, but they do rotate in the internal space with frequency $\sqrt{3}$.

Third, a standing wave is, by definition, the interference of two counter-propagating traveling waves with equal momenta. In our solution, one direction out of the two possible is selected for each branch; there is no counter-propagating partner, there is nothing to superpose — no standing node forms.

Thus, the correct description is: **a traveling wave with helical internal polarization**. The charged component rotates in the internal space (in the plane perpendicular to $(1,1,1)$) with period $\pi/\sqrt{3}$, and at the same time the entire configuration moves along the lightlike direction. The composition of "rotation plus translation" is precisely what yields the helix written out above. The analogy is exact: in circularly polarized light, the field vector rotates while the wave travels, and nobody calls the wave standing on the grounds that along the ray the vector merely rotates without

propagating. The only difference is that here the "polarization" lives not in a plane transverse to the ray in ordinary space, but in the internal matrix space.

An additional confirmation of the coherence of the traveling picture is provided by the phases of the three waves:

$$2s, \qquad -s + \sqrt{3}\,t, \qquad -s - \sqrt{3}\,t.$$

Their sum vanishes identically — this is the same condition $k_0 + k_+ + k_- = (0, 0)$. The product of the three phase factors equals

unity at every point of spacetime. In other words, the three traveling waves are phase-locked into a resonant triad: each moves along its own direction, but their phases are closed onto one another, and it is this closure that makes the combined object a unitary group rather than an arbitrary superposition.

Let us emphasize the strict status of this fact: **only the complete triad of modes forms a unitary configuration**. Each branch taken separately is an admissible plane wave, yet the unitarity condition

$$U^{\dagger}U = I$$

holds only when all three branches are present simultaneously with the exact projector amplitudes. Indeed, for a single branch $U_\alpha = e^{i\varphi_\alpha} P_\alpha$ one has

$$U_\alpha^\dagger U_\alpha = P_\alpha \neq I,$$

since no projector alone equals the identity matrix; it is only the sum $P_0 + P_+ + P_- = I$ that closes the condition. The phase-resonance condition $k_0 + k_+ + k_- = (0,0)$ binds the triad mutually:

removing any branch breaks the phase identity and destroys unitarity. Thus the phase locking is not a means of stabilizing one of the modes, but the very form in which the consistent solution exists: without $P_{\pm}$ the matrix U ceases to be unitary, and the helix loses its axis of rotation.

Fate of the Tachyonic Branch

In the isolated Klein–Gordon equation, the branch P_{-} appears tachyonic:

$$m_-^2 = -4\sqrt{3} < 0.$$

The mechanism of tachyonic instability, however, requires a homogeneous mode $k = 0$ — a zero-momentum wave growing exponentially without oscillation. The Helmholtz constraint rules out such a mode: all admissible momenta lie on the circle

$$k_x^2 + k_y^2 = 8,$$

and in particular, for the branch P_-,

$$|k_-|^2 = (\sqrt{3} - 1)^2 + (\sqrt{3} + 1)^2 = 8.$$

The branch has no homogeneous mode — the instability to which it would be entitled simply has nowhere to materialize. The actual frequency is positive:

$$\omega_- = \sqrt{3} - 1 \approx 0.73 > 0,$$

consistent with the dispersion relation:

$$\omega_-^2 - |k_-|^2 = (\sqrt{3} - 1)^2 - (\sqrt{3} + 1)^2 = -4\sqrt{3}.$$

The sign of m^2 has not disappeared; it has been reclassified: it characterizes the dispersion of a fixed point of the momentum

lattice, not a growing mode. The contribution of the branch to the Hamiltonian is positive:

$$H = \sum_{\text{modes}} \omega\left(a^\dagger a + \tfrac{1}{2}\right), \quad \omega_- > 0.$$

The charge under $Q = -iS/\sqrt{3}$:

$$q(P_-) = -1,$$

that is, P_- is the antiparticle of the branch P_+. Ordered by gaps, the spectrum reads

$$\omega_- = \sqrt{3} - 1 \approx 0.73 < \omega_0 = 2 < \omega_+ = \sqrt{3} + 1 \approx 2.73.$$

The former tachyon is the lightest charged particle of the model, three times lighter than the massive branch. The formal group velocity remains superluminal:

$$v_g = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} \approx 3.73,$$

but it carries no signal: two discrete points do not form a wave packet and have no front. The causal structure is set by the neutral branch P_0.

The physical summary can be phrased as follows. In an ordinary theory, a tachyon is a ball at the top of a hill: any perturbation grows. In our construction the hill exists, but one cannot get onto it: the homogeneous mode is excluded by the Helmholtz equation. The branch is not "suppressed from outside" — it could never have been unstable in the first place, since only states satisfying both constraints are admitted. What remains of the instability is a memory imprinted in the dispersion law — and the fact that this is the most accessible charged mode of the spectrum.

10. Quantization

The theory is quadratic, hence quantization is complete and exact. The positive-frequency selection yields 6 complex modes (3 branches × 2 momentum directions), each splitting into particle and antiparticle oscillators:

$$12 \text{ real oscillators.}$$

The Hamiltonian is positive:

$$H = \sum_{\text{modes}} \omega\left(a^\dagger a + \tfrac{1}{2}\right), \quad \omega > 0 \text{ for all branches.}$$

The vacuum zero-point energy:

$$E_0 = \tfrac{1}{2} \cdot 2\big(2 + (\sqrt{3}+1) + (\sqrt{3}-1)\big) = 2 + 2\sqrt{3}.$$

Charges under $Q = -iS/\sqrt{3}$:

$$q(P_0) = 0, \quad q(P_+) = +1, \quad q(P_-) = -1.$$

The branches P_+ and P_- are charge-conjugate; the neutral branch is self-conjugate. There are no ghosts, and the vacuum is stable: the tachyonic branch, confined by the Helmholtz constraint, is an ordinary stable particle of the spectrum.

11. Consistency Checks

For independent verification, at any point (x, y):

$$\operatorname{tr}\left(U''_{xx}U^{-1}\right) = \operatorname{tr}(-4I + 2iS) = -12,$$

$$\operatorname{tr}\left(U''_{yy}U^{-1}\right) = \operatorname{tr}(-4I - 2iS) = -12,$$

$$\operatorname{tr}\left(U''_{xy}U^{-1}\right) = \operatorname{tr}(-2W) = 0,$$

$$\operatorname{tr} S = 0, \quad \operatorname{tr} W = 0.$$

Conclusion

The following is strictly proven:

- the solution $U = e^{iW(x+y)+S(y-x)}$ exists, is written explicitly via projectors, and is unitary; both matrices arise from three Hermitian-unitary permutation matrices M_1, M_2, M_3 as anticommutators and commutators;
- it satisfies simultaneously the Helmholtz equation, the Klein–Gordon equation, and the mixed equation;

- in light-cone coordinates $s = x + y$, $t = x - y$, the system decomposes into the three equations $U''_{ss} = -W^2 U$, $U''_{tt} = S^2 U$, $U''_{st} = iSU$ and reduces entirely to the algebra of the matrices W and S;
- the solution space is finite-dimensional: 12 complex modes on a discrete momentum lattice;
- the system is neutrally stable with a finite margin $4 - 2\sqrt{3}$; the tachyonic band is suppressed by the constraint;

- causality is secured by the neutral massless branch lying exactly on the light cone;
- the tachyonic branch does not vanish but becomes a stable particle: gap $\sqrt{3}-1$, charge -1, antiparticle of the branch P_+;
- quantization is carried out exactly: positive Hamiltonian, 12 oscillators, stable vacuum with energy $2+2\sqrt{3}$, charges $0, \pm 1$.

All the listed results are theorems of direct computation; interpretations (the traveling helical wave, the $(3,3)$ signature, condensate extensions) are deliberately excluded from the paper, since they have the status of a consistent interpretation rather than a proven statement.

Appendix A. Orthogonality, Dimensions and Indecomposability of the Triad

A.1. Setup

The matrix

$$S = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \left[(1,1,1) \times (\cdot)\right]$$

is real and fully skew-symmetric: $S^T = -S$, with zero diagonal.

Hence automatically $S^\dagger = -S$ and $SS^T = S^T S$ — the matrix is normal. The spectrum:

$$\sigma(S) = \{\, 0,\ +i\sqrt{3},\ -i\sqrt{3}\,\},$$

with eigenvectors v_0, v_+, v_- of A.2. A useful identity:

$$S^2 = -3\,(I - P_0),$$

$$S^3 = -3S,$$

that is, the action is trivial along the axis $(1,1,1)$, while in the plane

$\Pi \perp (1,1,1)$ the exponential e^{-St} is an exact rotation with frequency $\sqrt{3}$ and period $\pi/\sqrt{3}$. The charge operator $Q = -iS/\sqrt{3}$ is Hermitian with eigenvalues $0, \pm 1$. The spectral projectors

$$P_\alpha = v_\alpha v_\alpha^\dagger,$$

$$\alpha \in \{0, +, -\}$$

satisfy the pairwise orthogonality and completeness conditions:

$$P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha,$$

$$P_0 + P_+ + P_- = I.$$

The purpose of the appendix is to clarify in precisely what sense the triad of modes is locked together, and to describe its geometry: the dimensions of the carrying spaces, the topology of the trajectory, and the character of the orthogonality.

A.2. Decomposition into real and imaginary parts

The eigenvectors

$$v_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$v_+ = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix},$$

$$v_- = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix},$$

$$\omega = -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}.$$

decompose into real and imaginary parts as follows:

$$\operatorname{Re} v_+ = \operatorname{Re} v_- = \frac{1}{\sqrt{3}}\left(1,\ -\tfrac{1}{2},\ -\tfrac{1}{2}\right),$$

$$\operatorname{Im} v_+ = -\operatorname{Im} v_- = \frac{1}{\sqrt{3}}\left(0,\ -\tfrac{\sqrt{3}}{2},\ +\tfrac{\sqrt{3}}{2}\right).$$

The real parts of the charged vectors coincide, the imaginary parts are opposite. Inner products part by part:

$$\operatorname{Re} v_+ \cdot \operatorname{Re} v_- = +\tfrac{1}{2},$$

$$\operatorname{Im} v_+ \cdot \operatorname{Im} v_- = -\tfrac{1}{2}.$$

Neither the real nor the imaginary halves are orthogonal on their own.

A.3. Hermitian orthogonality as balance

The Hermitian inner product adds the contributions of both halves:

$$\langle v_+, v_- \rangle = \operatorname{Re} v_+ \cdot \operatorname{Re} v_- + \operatorname{Im} v_+ \cdot \operatorname{Im} v_- = \tfrac{1}{2} - \tfrac{1}{2} = 0.$$

The orthogonality of the charged branches is thus a property of balance: the positive contribution of the real overlap is exactly compensated by the negative contribution of the imaginary part. Proof by contradiction: if the imaginary halves were orthogonal, then $\operatorname{Im} v_+ \cdot \operatorname{Im} v_- = 0$ and $\langle v_+, v_- \rangle = +\frac{1}{2} \neq 0$ — the orthogonality would be destroyed. Consequently, the orthogonality lives in neither half alone; it exists only in their coordinated balance.

For completeness: $\langle v_0, v_\pm \rangle = 0$ trivially, since $\operatorname{Re} v_\pm \perp (1,1,1)$ and $\operatorname{Im} v_\pm \perp (1,1,1)$ by construction.

A.4. Geometry: one plane, opposite orientations

Each complex line $\mathbb{C} \cdot v_{\pm}$ spans one and the same real plane

$$\Pi = \{\, u \in \mathbb{R}^3 : u \perp (1,1,1) \,\},$$

equipped with opposite orientations. The evolution of the charged components

$$e^{-i\sqrt{3}t},$$

$$e^{+i\sqrt{3}t}$$

is a rotation in Π with one and the same frequency $|\sqrt{3}|$, but in opposite directions — by analogy with a pair of counter-rotating circular polarizations. The identity $S^2 = -3\,(I - P_0)$ from A.1 makes this statement rigorous: on Π the operator S acts as a 90° rotation, and the exponential e^{-St} defines an exact rotation. The strict fact $P_+P_- = 0$ is compatible with the coincidence of the real carriers: the orthogonality is fixed by the Hermitian metric, not by the separation of planes.

A.5. Dimensional structure

The chain of inclusions of the carrying spaces:

- The full matrix space: $\mathrm{M}_3(\mathbb{C})$, complex dimension 9.
- The solution commutes with S, hence lives in its commutant — matrices diagonal in the Fourier basis:

$$U = \mathrm{diag}(U_0, U_+, U_-) \cong \mathbb{C}^3,$$

complex dimension 3, real dimension 6: $\mathbb{R}^6 \cong \mathbb{C}^3$. Each mode occupies a complex line $\mathbb{C} \cdot v_\alpha$, and the three lines together exhaust $\mathbb{C}^3$ — this is the resolution of unity from A.1.

- Unitarity cuts $\mathbb{C}^3$ down to the phase torus

$$U(1)^3 = \{\, (e^{i\varphi_0},\, e^{i\varphi_+},\, e^{i\varphi_-}) \,\},$$

real dimension 3; each mode lives on its own circle $U(1)$.

A.6. The phase torus and dense winding

The phases of the solution

$$\varphi_0 = 2s,$$

$$\varphi_+ = -s + \sqrt{3}\,t,$$

$$\varphi_- = -s - \sqrt{3}\,t$$

satisfy the identity

$$\varphi_0 + \varphi_+ + \varphi_- \equiv 0 \pmod{2\pi},$$

which is the pointwise form of the resonance condition $k_0 + k_+ + k_- = (0,0)$. One independent real constraint cuts the torus $U(1)^3$ down to the two-dimensional subtorus

$$T^2 = \{\, (\varphi_0, \varphi_+, \varphi_-) \in U(1)^3 \colon \varphi_0 + \varphi_+ + \varphi_- \equiv 0 \pmod{2\pi} \,\},$$

and it is along T^2 that the solution moves: the phases depend on two variables (s,t) — the two degrees of freedom of the torus.

Along the lines $x + y = \mathrm{const}$ the winding is one-parameter. The closure condition of the trajectory is the existence of integers

$(m, n) \neq (0, 0)$ such that the phase increment lies in $2\pi\mathbb{Z}^3$:

$$m\,(2, -1, -1) + n\,(0, \sqrt{3}, -\sqrt{3}) \in 2\pi\mathbb{Z}^3.$$

The second and third coordinates give $n\sqrt{3} = 2\pi k$ — impossible for integer $n \neq 0$, since $\sqrt{3}$ is irrational. The first coordinate then requires $2m \in 2\pi\mathbb{Z}$ for integer m — likewise impossible. Closure fails for every combination (m, n). The winding frequency ratio is $1 : \sqrt{3}$, irrational; by Kronecker's theorem the one-parameter trajectory is dense everywhere on T^2.

Conclusion: the solution does not wind onto a closed loop; it combs the entire phase torus uniformly and without repetitions. The helix in $\mathbb{R}^6$ of Section 9 is a section of this winding along $x + y = \text{const}$.

A.7. Indecomposability of the triad as a mathematical fact

The strict status of the locked structure: **only the complete triad of modes forms a unitary configuration**. Each branch separately

is an admissible plane wave $U_\alpha = e^{i\varphi_\alpha} P_\alpha$, yet

$$U_\alpha^\dagger U_\alpha = P_\alpha \neq I,$$

since no projector equals the identity by itself. The unitarity condition $U^\dagger U = I$ closes only via the full sum $P_0 + P_+ + P_- = I$ with exact amplitudes of all three projectors. Removing any branch violates both the phase identity of A.6 and the resolution of unity — the configuration ceases to exist as an element of the group.

Hence the precise meaning of the locked structure: the three

modes are **indecomposable as a mathematical structure** — the unitary configuration, the phase torus and the helical trajectory exist only with all three branches jointly present. This statement is purely structural: it concerns the algebraic make-up of the solution, not dynamical stabilization — within the linear model the branches evolve independently, and no feedback between them is required. The locked structure is a form of existence of a consistent solution, not a mechanism of interaction.

A.8. Differential identities and the double Schrödinger equation

The solution is written as

$$U = e^{iWs - St},$$

where $s = x + y, t = x - y$. The operators W and S commute and satisfy

$$WS = SW = -S,$$

$$-W^2 + S^2 = -4I.$$

First derivatives.

$$\frac{\partial U}{\partial x} = iWU - SU,$$

$$\frac{\partial U}{\partial y} = iWU + SU,$$

$$\frac{\partial U}{\partial x} - \frac{\partial U}{\partial y} = -2SU.$$

Second derivatives. Squaring the operators with due account of non-commutativity inside the square:

$$\frac{\partial^2 U}{\partial x^2} = (iW - S)^2 U = \bigl(-W^2 + S^2 - i(WS + SW)\bigr) U,$$

$$\frac{\partial^2 U}{\partial y^2} = (iW + S)^2 U = \bigl(-W^2 + S^2 + i(WS + SW)\bigr) U.$$

Substituting $WS + SW = -2S$ and $-W^2 + S^2 = -4I$:

$$\frac{\partial^2 U}{\partial x^2} = (-4I + 2iS)U,$$

$$\frac{\partial^2 U}{\partial y^2} = (-4I - 2iS)U.$$

Key identity.

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} = 4iSU,$$

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} = -2i\left(\frac{\partial U}{\partial x} - \frac{\partial U}{\partial y}\right).$$

The right-hand side contains the Hermitian operator iS, not the anti-Hermitian matrix S — this is exactly what guarantees the correctness of the writing.

Light-cone variables. In the variables s, t:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial s} + \frac{\partial}{\partial t},$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial s} - \frac{\partial}{\partial t},$$

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial s\,\partial t}.$$

The mixed derivative:

$$\frac{\partial^2 U}{\partial s\,\partial t} = iSU.$$

The double Schrödinger equation. From the identities above:

$$\frac{\partial^2 U}{\partial s\,\partial t} = -i\,\frac{\partial U}{\partial t},$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial s} + i\right)U = 0.$$

This equation splits into a pair of simultaneous Schrödinger equations:

$$i\,\frac{\partial U}{\partial s} = -W\,U,$$

$$i\,\frac{\partial U}{\partial t} = -iS\,U,$$

with two commuting "Hamiltonians": $-W$ with eigenvalues -2 (neutral mode) and $+1, +1$ (charged), and $-iS$ with eigenvalues $0, \pm\sqrt{3}$.

Structure of the general solution. The equation $\partial_t(\partial_s + i)U = 0$ has the general solution

$$U(s,t) = f(s) + e^{-is}\, h(t),$$

that is, the neutral component depends on s alone, while the charged factors appear as e^{-is} times an arbitrary function of t — precisely $\varphi_\pm = -s \pm \sqrt{3}\, t$. Unitarity selects exactly three modes from this space.

Factorized wave form. An equivalent compact writing:

$$(\partial_x + i)^2 U = (\partial_y + i)^2 U.$$

It reflects the symmetry of the problem under the swap $x \leftrightarrow y$ and is consistent with the commutation of W, S and the structure of the exponential.