```latex
\documentclass[pdftex,twocolumn,10pt,letterpaper]{article}
\usepackage[utf8]{inputenc}
\usepackage[english]{babel}
\usepackage{amsmath,amssymb,amsfonts}
\usepackage{graphicx}
\usepackage{bm}

\title{\textbf{Mathematical Modeling of Hydrodynamic Flows in the Vicinity of
Quasi-Ruled Surfaces of Negative Curvature and the Topology of Turbulent
Transition}}
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\begin{document}

\maketitle

\begin{abstract}
This paper investigates an original approach to the geometrization of the
kinematic characteristics of a continuous medium. A transition is made from
the classical linear differential continuity operator for an incompressible
fluid to a nonlinear functional-geometric consequence in the form of a
deformable hyperbolic paraboloid. A rigorous qualitative analysis of the
Cauchy problem for the Navier--Stokes equations, modified by the resulting
constraint in the local vicinity of a fixed point, is carried out. The
dynamics of the topological transition of a family of surfaces under a
changing time parameter of evolution is considered, which allows describing
the mechanism of transition from laminar to turbulent flow through geometric
bifurcations.
\end{abstract}

\section{Introduction and Model Formulation}
Consider an unsteady flow of an incompressible viscous fluid in a three-
dimensional Euclidean space. The classical system of Navier--Stokes equations
includes the dynamic equations of motion and the linear law of conservation
of mass (continuity equation):
\begin{equation}
\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} +
\frac{\partial v_3}{\partial x_3} = 0
\end{equation}

The aim of this study is to parameterize the components of the strain rate
tensor by introducing a spatial-phase argument $\varphi$:
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\begin{align}
\frac{\partial v_1}{\partial x_1} &= \varkappa \cdot \cos\varphi \nonumber \\
\frac{\partial v_2}{\partial x_2} &= \varkappa \cdot \cos\left(\varphi +
\frac{2\pi}{3}\right) \\
\frac{\partial v_3}{\partial x_3} &= \varkappa \cdot \cos\left(\varphi -
\frac{2\pi}{3}\right) \nonumber
\end{align}

Representation (2) trivially satisfies relation (1) due to the trigonometric
identity for shifted phases:
\begin{equation}
\varkappa \cdot \left[\cos\varphi + \cos\left(\varphi + \frac{2\pi}{3}\right)
+ \cos\left(\varphi - \frac{2\pi}{3}\right)\right] = 0
\end{equation}

Computing the difference of the deformation components along the second and
third coordinate axes, we obtain:
\begin{equation}
\frac{\partial v_3}{\partial x_3} - \frac{\partial v_2}{\partial x_2} =
\sqrt{3}\varkappa \cdot \sin\varphi
\end{equation}

Combining the equations, we eliminate the phase argument $\varphi$ by
transitioning to a quadratic form:
\begin{equation}
\left(\frac{\partial v_2}{\partial x_2} - \frac{\partial v_3}{\partial
x_3}\right)^2 \cdot \cos^2\varphi - 3 \cdot \left(\frac{\partial
v_1}{\partial x_1}\right)^2 \cdot \sin^2\varphi = 0
\end{equation}

Normalizing expression (5) by the magnitude of the local acceleration vector
of the medium $\frac{\partial |v|}{\partial t}$, we match it with the
structure of the classical Euler formula for the normal curvature of
surfaces. This isomorphism allows us to determine the principal curvatures of
the associated virtual surface:
\begin{align}
\kappa_1 &= \left(\frac{1}{\frac{\partial |v|}{\partial t}}\right) \cdot
\left(\frac{\partial v_2}{\partial x_2} - \frac{\partial v_3}{\partial
x_3}\right)^2 \nonumber \\
\kappa_2 &= \left(-\frac{3}{\frac{\partial |v|}{\partial t}}\right) \cdot
\left(\frac{\partial v_1}{\partial x_1}\right)^2
\end{align}
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Integrating the local principal curvatures (6) into the second-order
approximation equations (Monge form), we obtain a nonlinear equation of a
quasi-ruled surface (a dynamic deformable hyperbolic paraboloid), which is a
strict consequence of the flow kinematics.

\section{Analysis of the Cauchy Problem for the Modified System at Point M}
Let us introduce a local Cartesian coordinate system $(x, y, z)$, aligning
the origin $(0,0,0)$ with the studied point of the medium $M$ (where for
simplicity of notation we set $x_1=x, x_2=y, x_3=z$). It is important to
emphasize that the classical linear continuity equation represents a
fundamental physical law. However, to study local geometric instabilities and
spatial-phase constraints, we introduce a theoretical mathematical ansatz.
Instead of the classical equation, we require the fulfillment of a
functional-geometric constraint throughout the neighborhood of point $M$:
\begin{equation}
y^2 \cdot \left(\frac{\partial v_2}{\partial y} - \frac{\partial
v_3}{\partial z}\right)^2 - x^2 \cdot \left(\frac{\partial v_1}{\partial
x}\right)^2 - 2z \cdot \frac{\partial |v|}{\partial t} = 0
\end{equation}

Thus, the investigated modified Cauchy problem for the Navier--Stokes
equations takes the form of an idealized theoretical system where the Navier-
-Stokes equations of motion are combined with the local geometric constraint
(7) and the initial conditions for the velocity at time $t = 0$. This
localized model serves as an auxiliary analytical tool to investigate the
topology of the flow in the immediate vicinity of a singular point, without
violating the global integral conservation laws of the continuum.

\subsection{Dimensional Homogeneity Analysis}
Compliance with the principle of dimensional homogeneity is critically
important to substantiate the correctness of the model. The principal
curvatures of a surface have the dimension $\text{m}^{-1}$. The spatial
derivatives of the velocity field have the dimension $\text{s}^{-1}$, and the
local acceleration has $\text{m} \cdot \text{s}^{-2}$. Dimensional analysis
of the right-hand side of relation (6) yields:
\begin{equation}
\left[\left(\frac{1}{\frac{\partial |v|}{\partial t}}\right) \cdot
\left(\frac{\partial v_i}{\partial x_j}\right)^2\right] =
\left(\frac{1}{\text{m} \cdot \text{s}^{-2}}\right) \cdot (\text{s}^{-1})^2 =
\text{m}^{-1}
\end{equation}

Which formally confirms homogeneity. To strictly eliminate scale
contradictions, a characteristic spatial scale of local flow inhomogeneity
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$L_0$ is introduced, transforming the spatial basis into a dimensionless
form.

\subsection{Mathematical Features of Local Degeneracy}
The transition to the nonlinear constraint (7) entails a radical
transformation of the topological properties of the Navier--Stokes equations:
\begin{enumerate}
    \item \textit{Disappearance of the constraint at the origin:} Directly at
the center of the local system (point $M$), where $x=0, y=0, z=0$, equation
(7) degenerates into a trivial identity $0=0$. As a consequence, at the fixed
point $M$, the hydrodynamic system becomes locally underdetermined (3
equations of motion for 4 unknown functions of velocity and pressure).
    \item \textit{Change in the type of the pressure operator:} In the
traditional formulation, the divergence operator reduces the system to an
elliptic Poisson equation for pressure. Differentiating the nonlinear
equation (7) with respect to the time variable and substituting the
accelerations from the equations of motion leads to a differential equation
for the pressure field containing a linear combination of the first spatial
derivatives $\frac{\partial p}{\partial x_i}$ with a weighting factor $z$.
This indicates a transition of the system from the parabolic-elliptic class
to the class of degenerating nonlinear systems of mixed (hyperbolic) type.
\end{enumerate}

\section{Exact Solutions for Planar Parallel Shear Flow ($v_3 = 0$)}
To study the analytical properties of the modified system, let us consider
the case of a planar parallel shear flow, where the velocity component $v_3 =
0$, and the quantities $v_1, v_2$ are invariant with respect to the $z$
coordinate ($\partial/\partial z = 0$). The constraint equation (7) reduces
to the form:
\begin{equation}
y^2 \cdot \left(\frac{\partial v_2}{\partial y}\right)^2 - x^2 \cdot
\left(\frac{\partial v_1}{\partial x}\right)^2 - 2z \cdot \frac{\partial
\sqrt{v_1^2+v_2^2}}{\partial t} = 0
\end{equation}

Since the velocity components cannot depend on the $z$ coordinate, the
explicit presence of the linear factor $z$ before the unsteady term requires
separation of variables, forming two physically distinct flow regimes.

\subsection{Steady-state Regime in Terms of Velocity Magnitude
($\frac{\partial |v|}{\partial t} = 0$)}
In the case where the absolute value of the velocity is constant over time,
the time term vanishes, transforming the constraint into a purely geometric
form:
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\begin{equation}
\left|y \cdot \frac{\partial v_2}{\partial y}\right| = \left|x \cdot
\frac{\partial v_1}{\partial x}\right|
\end{equation}

Integration of equation (9) by the method of separation of variables allows
obtaining a self-similar class of solutions in the form of power functions of
deformation, where the velocity profiles are expressed through power-law
dependencies of the coordinates. The general solution of equation (9) on the
entire $Oxy$ plane is a logarithmic velocity profile:
\begin{align}
v_1(x, y) &= A_1 \cdot \ln|x| + \psi(y) \nonumber \\
v_2(x, y) &= A_2 \cdot \ln|y| + \omega(x), \quad \text{provided } |A_1| =
|A_2|
\end{align}

\subsection{Unsteady Regime ($\frac{\partial |v|}{\partial t} \neq 0$)}
If the velocity magnitude is unsteady, the explicit linear dependence of
equation (8) on the $z$ coordinate contradicts the condition of planar shear.
The only mathematically compatible condition for the existence of such a flow
in the entire neighborhood of point $M$ is the identical zeroing of both
blocks of the equation:
\begin{text}
$\frac{\partial |v|}{\partial t} = 0$ \quad and \quad $y^2 \cdot
\left(\frac{\partial v_2}{\partial y}\right)^2 - x^2 \cdot
\left(\frac{\partial v_1}{\partial x}\right)^2 = 0$
\end{text}

This proves the structural degeneracy theorem: the modified consequence of
the continuity equation completely bans dynamic fluctuations of the absolute
velocity magnitude in a planar parallel flow. Any evolution in time can be
associated exclusively with a change in the direction of the velocity vector
(vortical rotation), but not with a change in the kinetic energy of the local
volume of a fluid particle.

\section{Dynamics of Topological Transition and Turbulence}
To describe the evolution of the flow over time, let us consider a
generalized family of surfaces depending on the dynamic deformation parameter
$\varepsilon = \varepsilon(t)$:
\begin{equation}
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z + \varepsilon \cdot z^2
\end{equation}
Geometric analysis of equation (12) reveals three qualitative states of the
medium depending on the sign of the evolution parameter:
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\begin{itemize}
\item At $\varepsilon = 0$: the surface represents a classical hyperbolic
paraboloid (laminar shear regime).
\item At $\varepsilon > 0$: after completing the square with respect to the
$z$ coordinate, the equation transforms into a one-sheeted hyperboloid.
\item At $\varepsilon < 0$: the topology of the medium transforms into a two-
sheeted hyperboloid.
\end{itemize}
\subsection{Derivation of the Differential Equation of Geometric Evolution}
Suppose that over a small time interval $\Delta t$, the initial state of the
paraboloid (12) transitions into a new family with a changed parameter:
\begin{equation}
\frac{x^2}{a^2} + \varepsilon \cdot \frac{z^2}{c^2} - \frac{y^2}{b^2} =
\varepsilon
\end{equation}
Expressing the value of $\varepsilon(t)$ from the initial state (12) and the
value of $\varepsilon(t + \Delta t)$ from the final state (13), we find their
increment $\Delta \varepsilon = \varepsilon(t + \Delta t) - \varepsilon(t)$.
Passing to the differential form $\partial\varepsilon/\partial t \approx
\Delta\varepsilon/\Delta t$, we obtain the equation for the rate of evolution
of the flow geometry:
\begin{equation}
\frac{\partial \varepsilon}{\partial t} = \left(\frac{1}{\Delta t}\right)
\cdot \left[\frac{c^2 \cdot \left(\frac{y^2}{a^2} -
\frac{x^2}{b^2}\right)}{a^2 b^2 \cdot (z^2 - c^2)} - \varepsilon\right]
\end{equation}
Integrating equation (14) according to a linear law, we obtain the implicit
equation of the moving surface $F(x,y,z,t) = 0$:
\begin{equation}
\frac{x^2}{a^2} - \frac{y^2}{b^2} - 2z - z^2 \cdot \left[\varepsilon_0 +
\left(\frac{t}{\Delta t}\right) \cdot \left(\frac{c^2 \cdot
\left(\frac{y^2}{a^2} - \frac{x^2}{b^2}\right)}{a^2 b^2 \cdot (z^2 - c^2)} -
\varepsilon_0\right)\right] = 0
\end{equation}
\subsection{Transition to Principal Curvatures and Physical Meaning}
Let us rename the geometric constants in terms of the magnitudes of the
principal curvatures of the flow: $b^2 = \frac{c}{|k_1|}$ and $a^2 =
\frac{c}{|k_2|}$. Then the final state of the deformed surface takes the
form:
\begin{equation}
F(x, y, z, t + \Delta t) = |k_2|x^2 + \varepsilon \cdot \frac{z^2}{c} -
|k_1|y^2 - \frac{c}{\varepsilon} = 0
\end{equation}
Analysis of the difference between the states shows:
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\begin{enumerate}
\item In the process of evolution ($0 < t < \Delta t$), due to the nonlinear
entry of the $z^2$ term, the surface begins to intensely twist and deform.
When passing through critical points of the sign change, a flow bifurcation
occurs.
\item At time $t = \Delta t$, the linear term $2z$ is completely compensated,
and the entire structure transforms into a cone or a conic section.
\end{enumerate}
\textbf{Physical conclusion:} The described continuous topological transition
from a smooth paraboloid (``saddle'') to a twisted one-sheeted hyperboloid
and conical collapse strictly models on the macrolevel the process of
transition from laminar to turbulent flow. Local turbulent vortex tubes
originate precisely at the moments of spatial bifurcations of the velocity
field geometry.
\section{Conclusion}
An integrated nonlinear model linking the Navier--Stokes equations with the
dynamic evolution of quasi-ruled surfaces is proposed. It is shown that
introducing the time parameter $\varepsilon(t)$ allows mathematically
rigorous description of the restructuring of a laminar hydrodynamic field
into a turbulent one through spatial bifurcations of the Monge form. The
resulting wave jumps and conic sections open new geometric methods for
analyzing the stability of viscous fluid flows.
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