# Hydrodynamic Singularity and Coherent Vortex Structures in Navier–Stokes Flows under a Local Non-Linear Continuity Constraint

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We investigate the asymptotic localized behavior of a modified three-dimensional Navier–Stokes system where the classical linear incompressibility condition $\nabla \cdot \mathbf{v} = 0$ is replaced by a non-linear algebraic functional relation $Q(x, y, z, t) = 0$. This surrogate relation acts as a convex combination of localized deformation characteristics weighted by parameters $a, b, c > 0$. Using a localized Taylor expansion near a degenerate flow point, we demonstrate that this non-linear constraint triggers a differential-algebraic evolution susceptible to a finite-time gradient blow-up. We identify a critical kinematic viscosity threshold $\nu_{\text{crit}}$ below which a spatial kink develops in the velocity profile. Crucially, a parameter resonance condition $\sqrt{a} + \sqrt{b} = 2\sqrt{c}$ is discovered, which acts as a stabilizing attractor that restores the classical divergence-free state and mitigates the blow-up singularity. Finally, the topology of coherent vortex structures under this non-linear constraint is mapped using a modified $Q$-invariant criterion, and global energy dissipation estimates are established.

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## I. INTRODUCTION

The classical Navier–Stokes equations govern the balance of linear momentum and mass conservation in viscous incompressible fluids. In standard formulations, mass conservation translates to the strict linear kinematic constraint $\nabla \cdot \mathbf{v} = 0$. However, in regions of intense localized shear, non-equilibrium states, or when modeling complex multi-scale fluids using surrogate macroscopic constraints, non-linear alternatives to the continuity equation emerge as vital tools for stability analysis and subgrid-scale parameterizations.

In this work, we diverge from the standard incompressibility assumption to explore the physical consequences of a non-linear surrogate continuity condition. Rather than viewing this condition merely as a mathematical perturbation, we treat it as a localized non-linear deformation field that accounts for anisotropic multi-axial strains. By performing a localized asymptotic analysis around a fixed degeneracy point in the flow, we examine how such non-linear constraints modify the convective-diffusive balance, influence the formation of hydrodynamic singularities (gradient blow-up), and alter the topology of coherent vortex structures.

## II. PROBLEM FORMULATION AND THE NON-LINEAR CONSTRAINT

In the paper *Mathematical Modeling of Hydrodynamic Flows in the Vicinity of Quasi-Ruled Surfaces of Negative Curvature and the Topology of Turbulent Transition*[1], the author introduces a spatial-phase parameterization of the strain-rate tensor and establishes an exact, localized nonlinear invariant that directly links fluid deformation to the evolution of physical surfaces possessing negative Gaussian curvature:

$$k_1 = \left(\frac{1}{\partial|\mathbf{v}|/\partial t}\right)\left(\frac{\partial v_2}{\partial x_2} - \frac{\partial v_3}{\partial x_3}\right)^2, \quad k_2 = \left(\frac{-3}{\partial|\mathbf{v}|/\partial t}\right)\left(\frac{\partial v_1}{\partial x_1}\right)^2 \tag{1}$$

In view of this, instead of the classical continuity equation, the author required the satisfaction of a functional-geometric constraint throughout the neighbor-

hood of the point within a local Cartesian coordinate system $(x, y, z)$ aligned with the origin $(0, 0, 0)$ at the point $M$ of the medium under consideration:

$$y^2 \left( \frac{\partial v_2}{\partial y} - \frac{\partial v_3}{\partial z} \right)^2 - x^2 \left( \frac{\partial v_1}{\partial x} \right)^2 - 2z \left( \frac{\partial |\mathbf{v}|}{\partial t} \right) = 0 \tag{2}$$

In this work, within the local Cartesian coordinate system $(x, y, z)$ centered at the flow's critical point $M(0, 0, 0)$, we replace the standard zero-divergence condition with a nonlinear constraint defined by a weighted combination of the constraints adopted in the paper *Mathematical Modeling of Hydrodynamic Flows in the Vicinity of Quasi-Ruled Surfaces of Negative Curvature and the Topology of Turbulent Transition*[1]:

$$Q(x, y, z, t) = aQ_1 + bQ_2 + cQ_3 = 0, \tag{3}$$

where the positive weight coefficients model the directional anisotropy of the fluid's volume-preservation response, satisfying:

$$a, b, c > 0, \quad a + b + c = 1. \tag{4}$$

The individual component functions $Q_i$ represent a coupling between the squared longitudinal strain rates and the local rate of change of the kinetic energy density (represented here via the velocity magnitude $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$):

$$Q_1 = y^2 \left( \frac{\partial v_2}{\partial y} - \frac{\partial v_3}{\partial z} \right)^2 - x^2 \left( \frac{\partial v_1}{\partial x} \right)^2 - 2z \frac{\partial |\mathbf{v}|}{\partial t} = 0, \tag{5}$$

$$Q_2 = z^2 \left( \frac{\partial v_3}{\partial z} - \frac{\partial v_1}{\partial x} \right)^2 - y^2 \left( \frac{\partial v_2}{\partial y} \right)^2 - 2x \frac{\partial |\mathbf{v}|}{\partial t} = 0, \tag{6}$$

$$Q_3 = x^2 \left( \frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y} \right)^2 - z^2 \left( \frac{\partial v_3}{\partial z} \right)^2 - 2y \frac{\partial |\mathbf{v}|}{\partial t} = 0. \tag{7}$$

Physically, Eqs. (5)–(7) establish a local feedback loop: any transient acceleration of the fluid magnitude at the origin generates a spatial gradient in the surrounding volume deformation field, heavily penalizing single-axis compressions.

## III. LOCAL KINEMATIC STATIONARITY AND CONVECTIVE BALANCE

Evaluating the system directly at the origin $M(0,0,0)$ reveals an important kinematic property. Because the quadratic spatial terms in Eqs. (5)–(7) vanish due to the coordinate factors $x^2, y^2, z^2$, the constraint equation (3) degenerates into a linear combination of spatial increments:

$$-2\frac{\partial|\mathbf{v}|}{\partial t}\cdot(b\cdot x + c\cdot y + a\cdot z) = 0. \tag{8}$$

For this relation to remain valid within an arbitrary, isotropic $\epsilon$-neighborhood of the origin, the coefficient representing the temporal variation of the velocity magnitude must vanish identically. Thus, we obtain a local stationarity condition for the velocity norm:

$$\left.\frac{\partial|\mathbf{v}|}{\partial t}\right|_M = 0 \implies |\mathbf{v}(0,0,0,t)| = \text{const.} \tag{9}$$

This local freezing of the kinetic energy density fundamentally rewrites the momentum balance at the critical point. Since $\frac{\partial|\mathbf{v}|}{\partial t} = (\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t})/|\mathbf{v}| = 0$, taking the scalar product of the Navier–Stokes momentum equation with the velocity vector $\mathbf{v}$ yields:

$$(\mathbf{v}\cdot\nabla p)|_M = \rho\nu(\mathbf{v}\cdot\Delta\mathbf{v})|_M - \rho(\mathbf{v}\cdot((\mathbf{v}\cdot\nabla)\mathbf{v}))|_M\,. \tag{10}$$

Equation (10) dictates that at the core of this non-linear interaction, the work done by the local pressure gradient is entirely balanced by the competition between convective inertial transport and viscous diffusion, free from any local temporal unsteadiness.

## IV. VISCOUS FAILURE THRESHOLD AND GRADIENT BLOW-UP

To understand how this non-linear continuity surrogate induces structural transitions in the flow, we expand the constraint equation to the next dominant spatial scale, $O(r^4)$. Grouping independent spatial coordinate coefficients yields a rigid algebraic system for the longitudinal strain components $d_1 = \frac{\partial v_1}{\partial x}$,

$d_2 = \frac{\partial v_2}{\partial y}$, and $d_3 = \frac{\partial v_3}{\partial z}$:

$$c(d_1 - d_2)^2 - ad_1^2 = 0, \quad a(d_2 - d_3)^2 - bd_2^2 = 0, \quad b(d_3 - d_1)^2 - cd_3^2 = 0. \tag{11}$$

Solving Eq. (11) links the strain rates through scaling metrics:

$$d_2 = \kappa_1 d_1, \quad d_3 = \kappa_2 d_1, \tag{12}$$

where $\kappa_1 = 1 \pm \sqrt{a/c}$ and $\kappa_2 = \kappa_1 \left(1 \pm \sqrt{b/a}\right)$.

By differentiating the $x$-momentum equation with respect to $x$ and utilizing the pressure Poisson equation, we derive a Riccati-type evolutionary equation for the primary strain rate $d_1(t)$:

$$\frac{\partial d_1}{\partial t} = -\Gamma_{abc} d_1^2 + \nu \Delta d_1 \approx -\Gamma_{abc} d_1^2 - \nu \frac{\Lambda_{abc}}{L^2} d_1, \tag{13}$$

where $L$ represents the characteristic localization scale of the flow, $\Gamma_{abc} = -2\left(\kappa_1 + \kappa_2 + \kappa_1/\kappa_2\right)$, and $\Lambda_{abc} = 1 + (1 + \kappa_1 + \kappa_2)^2$. Integrating this Bernoulli-type equation yields the analytical trajectory of the strain rate:

$$d_1(t) = \frac{d_1(0) e^{-\nu \frac{\Lambda_{abc}}{L^2} t}}{1 - \frac{\Gamma_{abc} L^2 d_1(0)}{\nu \Lambda_{abc}} \left(1 - e^{-\nu \frac{\Lambda_{abc}}{L^2} t}\right)}. \tag{14}$$

Equation (14) unveils a critical hydrodynamic phenomenon. If $\Gamma_{abc} < 0$ and the fluid undergoes initial compression ($d_1(0) < 0$), the denominator can vanish in finite time. This marks a finite-time gradient blow-up, resulting in a sharp spatial kink in the velocity profile. This catastrophic steepening occurs whenever the dissipative capacity of the fluid fails to match the non-linear amplification, establishing a critical kinematic viscosity threshold:

$$\nu_{\text{crit}} = \frac{|\Gamma_{abc}|}{\Lambda_{abc}} |d_1(0)| L^2. \tag{15}$$

For regimes where $\nu \leq \nu_{\text{crit}}$, viscous diffusion cannot redistribute the localized compression fast enough, leading to a localized shock-like breakdown of the smooth velocity field.

## V. RESONANT SELF-STABILIZATION AND COHERENT VORTEX TOPOLOGY

A remarkable feature of this formulation is the existence of a parameter space that eliminates the blow-up singularity. The classical fluid incompressibility condition ($\operatorname{div}\mathbf{v} = d_1 + d_2 + d_3 = 0$) is satisfied if and only if the weight parameters satisfy the resonant relation:

$$\sqrt{a} + \sqrt{b} = 2\sqrt{c}. \tag{16}$$

When this structural resonance occurs, the non-linear coupling coefficient flips sign to become strictly positive ($\Gamma_{abc} > 0$). In this regime, the non-linear term in the Riccati equation (13) transforms from a destabilizing driver into a powerful damping mechanism, providing complete self-stabilization of the flow across all initial conditions.

To evaluate how this affects the organization of the flow, we examine the second invariant of the velocity gradient tensor, $Q = -\frac{1}{2}\mathrm{tr}(\mathbf{G}^2)$, which is widely utilized in the fluid dynamics community (Hunt's criterion[2]) to isolate coherent vortex cores. Decomposing the gradient tensor $\mathbf{G}$ into the symmetric strain-rate tensor $\mathbf{S}$ and the anti-symmetric vorticity tensor $\mathbf{\Omega}$, and embedding our non-linear parameters, the modified criterion for identifying a coherent vortex at the origin becomes:

$$Q_M = \frac{1}{4}|\boldsymbol{\omega}|^2 - |S_{\mathrm{shear}}|^2 - \frac{1}{2}\Gamma_{abc}d_1^2 > 0, \tag{17}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the vorticity vector, and $\mathbf{S}_{\mathrm{shear}}$ isolates the pure out-of-plane shear deformation. Equation (17) indicates that the non-linear continuity surrogate elevates the threshold required for vortex identification. A coherent vortex structure is registered only if the rotational vorticity overcomes not just the standard fluid shearing, but also an additional energy penalty proportional to the localized longitudinal compression squared ($\frac{1}{2}\Gamma_{abc}d_1^2$).

## VI. GLOBAL INTEGRAL CLOSURE AND ENERGY DISSIPATION

To confirm that this localized modification remains physically consistent at macro-scales, we integrate the system over the full free-space domain $\mathbb{R}^3$, imposing decaying Dirichlet boundary conditions at infinity ($\lim_{|\mathbf{x}|\to\infty} \mathbf{v} = 0$). Under the stabilizing resonance condition (16), the classical divergence-free behavior is recovered globally, enabling us to uniquely map the pressure field using the Newtonian potential:

$$p(\mathbf{x},t) = p_\infty + \frac{\rho}{4\pi}\int_{\mathbb{R}^3} \frac{\sum_{i,j} \frac{\partial v_j}{\partial x_i'}(\mathbf{x}',t)\frac{\partial v_i}{\partial x_j'}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}'. \tag{18}$$

Taking the scalar product of the momentum equations with $\mathbf{v}$ and integrating over $\mathbb{R}^3$, the convective terms and pressure gradients vanish due to the global divergence-free condition and integration by parts. This provides a classical kinetic energy dissipation identity:

$$\frac{d}{dt}E_{\text{kin}}(t) = \frac{d}{dt}\left(\frac{\rho}{2}\int_{\mathbb{R}^3}|\mathbf{v}|^2 d\mathbf{x}\right) = -\rho\nu\int_{\mathbb{R}^3}|\nabla\mathbf{v}|^2 d\mathbf{x} \leq 0. \tag{19}$$

Equation (19) proves that despite the highly non-linear, non-local features acting near the degenerate zones, the global system obeys strict thermodynamic bounds, ensuring monotonic energy decay via viscous dissipation.

## VII. CONCLUSIONS

We have analyzed the localized physics of a 3D Navier–Stokes flow governed by a non-linear surrogate continuity condition. The substitution of the linear incompressibility constraint alters the system behavior, triggering a finite-time gradient blow-up when the fluid's viscosity drops below a critical threshold $\nu_{\text{crit}}$. However, when the parameters satisfy the resonant state $\sqrt{a} + \sqrt{b} = 2\sqrt{c}$, the system self-stabilizes, fully recovering the divergence-free flow fields and establishing global energy bounds. These insights offer an analytical framework for designing non-linear subgrid filters and stabilized numerical schemes for high-Reynolds-number hydrodynamics.

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