```
\documentclass[11pt,a4paper]{article}

\usepackage[utf8]{inputenc}

\usepackage[english]{babel}

\usepackage{amsmath,amssymb,amsfonts,amsthm}

\usepackage{geometry}

\geometry{margin=1in}


\title{\textbf{Analysis of the Cauchy Problem for the Navier--Stokes
Equations with a Modified Continuity Condition in a Local
Neighborhood}}

\author{\textbf{Manarbek Akilbayev} \\e-mail: makilbayev@mail.ru
\\textit{Department of Mathematics / Independent Researcher}}

\date{\small \11.09.2026}


\newtheorem{theorem}{Theorem}


\begin{document}


\maketitle


\begin{abstract}

In this paper, we perform an asymptotic, geometric, and integral
analysis of a modified Navier--Stokes system for a viscous
incompressible fluid in three-dimensional space. We consider a
deformation of the classical hydrodynamic system in which the linear
continuity equation $\nabla \cdot \mathbf{v} = 0$ is replaced by a
nonlinear algebraic functional relation $Q(x,y,z,t)=0$, constructed as
a convex combination with weight coefficients $a, b, c > 0$. The
research method is based on the local Taylor expansion of fields in
the neighborhood of a fixed degeneracy point aligned with the origin
of a local coordinate system. We prove theorems on local kinematic
stationarity of the velocity magnitude, the existence of a critical
viscous threshold for gradient blow-up, conditions for stabilizing
resonance of parameters, and the topology of coherent vortex
structures based on the $Q$-invariant criterion. An integral closure
of the problem in the space $\mathbb{R}^3$ is provided.

\end{abstract}


\section{Introduction and Problem Statement}

The classical system of Navier--Stokes equations describing the
dynamics of a viscous incompressible fluid consists of the vector
```

momentum balance equation and a linear kinematic constraint on the divergence of the velocity field. The present work is devoted to the study of the Cauchy problem for a modified evolutionary system, where the law of conservation of mass is replaced by a nonlinear surrogate constraint equation.

Let us consider the behavior of the hydrodynamic system everywhere in the neighborhood of a fixed point M. We align the origin of the local Cartesian coordinate system $(x,y,z) = (0,0,0)$ with the point M. Instead of the standard continuity equation, the constraint function is given by:

\begin{equation}

Q(x,y,z,t) = aQ_1 + bQ_2 + cQ_3 = 0

\end{equation}

where the weight coefficients satisfy the normalization and strict positivity conditions:

\begin{equation}

a, b, c > 0, \quad a + b + c = 1

\end{equation}

The partial nonlinear functions Q_i, which are differential consequences of the local deformation structure, are defined as:

\begin{equation}

Q_1 = y^2 \left(\frac{\partial v_2}{\partial y} - \frac{\partial v_3}{\partial z} \right)^2 - x^2 \left(\frac{\partial v_1}{\partial x} \right)^2 - 2z \frac{\partial |\mathbf{v}|}{\partial t} = 0

\end{equation}

\begin{equation}

Q_2 = z^2 \left(\frac{\partial v_3}{\partial z} - \frac{\partial v_1}{\partial x} \right)^2 - y^2 \left(\frac{\partial v_2}{\partial y} \right)^2 - 2x \frac{\partial |\mathbf{v}|}{\partial t} = 0

\end{equation}

\begin{equation}

Q_3 = x^2 \left(\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y} \right)^2 - z^2 \left(\frac{\partial v_3}{\partial z} \right)^2 - 2y \frac{\partial |\mathbf{v}|}{\partial t} = 0

\end{equation}

where $\mathbf{v} = (v_1, v_2, v_3)$ is the local velocity vector, and $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ is its Euclidean norm. The problem is closed by prescribing smooth initial conditions at $t=0$:

\begin{equation}

v_1(x,y,z,0) = v_1^0(x,y,z), \quad v_2(x,y,z,0) = v_2^0(x,y,z), \quad v_3(x,y,z,0) = v_3^0(x,y,z)

\end{equation}

\section{Main Theorems and Analytical Proofs}

\begin{theorem}[On Local Kinematic Degeneracy and Stationarity of the Velocity Magnitude]

Let the velocity vector-function $\mathbf{v}(x,y,z,t) \in C^2(\mathbb{R}^3 \times [0, T])$ satisfy the modified continuity equation (1). Then, at the origin of the local coordinate system $M(0,0,0)$, the local acceleration of the magnitude vanishes identically:

\begin{equation}

\left. \frac{\partial |\mathbf{v}|}{\partial t} \right|_M = 0 \implies |\mathbf{v}(0,0,0,t)| = \text{const}, \quad \forall t \in [0, T]

\end{equation}

Furthermore, the projection of the pressure gradient onto the direction of the local velocity is completely determined by the balance of convective transport and viscous diffusion:

\begin{equation}

\left. (\mathbf{v} \cdot \nabla p) \right|_M = \left. \rho \nu (\mathbf{v} \cdot \Delta \mathbf{v}) \right|_M - \left. \rho (\mathbf{v} \cdot ((\mathbf{v} \cdot \nabla)\mathbf{v})) \right|_M

\end{equation}

\end{theorem}

\begin{proof}

By assumption, at the point M, the spatial coordinates are identically zero: $x=0, y=0, z=0$. Expanding the full expression of the functional (1) and substituting zero values for the coordinates, we observe that all quadratic differential terms containing the factors x^2, y^2, z^2 vanish. The function Q degenerates into a linear form with respect to spatial increments:

\begin{equation}

-2 \frac{\partial |\mathbf{v}|}{\partial t} \cdot (b \cdot x + c \cdot y + a \cdot z) = 0

\end{equation}

This equality must hold in any small ϵ-neighborhood of the point M. Since the linear combination $(bx + cy + az)$ given strictly positive weights $a,b,c > 0$ defines a plane and does not equal zero identically under arbitrary spatial displacements $(x,y,z) \neq (0,0,0)$, the only continuous solution in the neighborhood of M is the vanishing of the first factor, which proves relation (7).

To derive the pressure equation (8), we expand the time derivative of the velocity magnitude using the scalar product:

\begin{equation}

\frac{\partial |\mathbf{v}|}{\partial t} = \frac{\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t}}{|\mathbf{v}|} = 0 \implies \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} = 0

\end{equation}

Multiplying the Navier--Stokes vector momentum equation scalarly by $\mathbf{v}$ at point M and substituting (10), we obtain the desired balance relation (8). Theorem 1 is proven.

\end{proof}

\begin{theorem}[On the Structure of the Gradient Tensor and the Critical Viscous Threshold]

In the leading spatial scale of the asymptotic expansion $O(r^4)$ in the neighborhood of $M(0,0,0)$, the longitudinal strain components of the velocity gradient tensor $d_1 = \frac{\partial v_1}{\partial x}$, $d_2 = \frac{\partial v_2}{\partial y}$, $d_3 = \frac{\partial v_3}{\partial z}$ are coupled by a rigid nonlinear algebraic system:

\begin{equation}

d_2 = \kappa_1 d_1, \quad d_3 = \kappa_2 d_1; \quad \kappa_1 = 1 \pm \sqrt{\frac{a}{c}}, \quad \kappa_2 = \kappa_1 \left(1 \pm \sqrt{\frac{b}{a}}\right)

\end{equation}

There exists a critical threshold of kinematic viscosity ν_{crit}, determined by the localization scale of the flow L:

\begin{equation}

\nu_{\text{crit}} = \frac{|\Gamma_{abc}|}{\Lambda_{abc}} |d_1(0)| L^2

\end{equation}

where $\Gamma_{abc} = -2\left(\kappa_1 + \kappa_2 + \frac{\kappa_1}{\kappa_2}\right)$ and $\Lambda_{abc} = 1 + (1 + \kappa_1 + \kappa_2)^2$. For $\nu \leq \nu_{\text{crit}}$ and $\Gamma_{abc} < 0$, the local solution breaks down in finite time with the formation of a spatial kink in the velocity profile (blow-up).

\end{theorem}

\begin{proof}

Since the linear terms in time compensate for each other by Theorem 1, the next significant scale of the constraint equation (1) involves terms of the fourth order of smallness in r. Grouping the coefficients for the independent squares of coordinates x^2, y^2, z^2, we obtain:

\begin{equation}

c(d_1 - d_2)^2 - a d_1^2 = 0, \quad a(d_2 - d_3)^2 - b d_2^2 = 0, \quad b(d_3 - d_1)^2 - c d_3^2 = 0

\end{equation}

Extracting the roots from these equations leads to the linear relations (11).

Applying the spatial derivative operator $\frac{\partial}{\partial x}$ to the x-component of the equations of motion and evaluating its value at point M, taking into account the pressure projection from the Poisson equation, we obtain a Riccati-type evolutionary equation for the component $d_1(t)$:

\begin{equation}

\frac{\partial d_1}{\partial t} = -\Gamma_{abc} d_1^2 + \nu \Delta d_1 \approx -\Gamma_{abc} d_1^2 - \nu \frac{\Lambda_{abc}}{L^2} d_1

\end{equation}

Integration of this Bernoulli equation yields the exact analytical solution:

\begin{equation}

d_1(t) = \frac{d_1(0) e^{-\nu \frac{\Lambda_{abc}}{L^2} t}}{1 - \frac{\Gamma_{abc} L^2 d_1(0)}{\nu \Lambda_{abc}} \left(1 - e^{-\nu \frac{\Lambda_{abc}}{L^2} t}\right)}

\end{equation}

The denominator of fraction (15) vanishes in finite time for $\Gamma_{abc} < 0$ and initial compression $d_1(0) < 0$ if and only if the stationary pre-exponential factor exceeds or equals unity. This directly imposes a restriction on the dissipative capacity of the medium and formulates the viscous failure criterion (12). Theorem 2 is proven.

\end{proof}

\begin{theorem}[On the Resonant Stabilizing Attractor and Vortex Topology]

The classical flow incompressibility condition $\text{div} \, \mathbf{v} = 0$ in the neighborhood of $M(0,0,0)$ is satisfied if and only if the weight coefficients of the constraint functional satisfy the resonant relation:

\begin{equation}

\sqrt{a} + \sqrt{b} = 2\sqrt{c}

\end{equation}

When condition (16) is met, the effective nonlinear coefficient becomes strictly positive ($\Gamma_{abc} > 0$), providing self-stabilization of the flow, and the criterion for identifying a coherent vortex (Q-invariant) at point M takes the form:

\begin{equation}

Q_M = \frac{1}{4} |\boldsymbol{\omega}|^2 - |S_{\text{shear}}|^2 - \frac{1}{2}\Gamma_{abc} d_1^2 > 0

\end{equation}

\end{theorem}

\begin{proof}

The incompressibility condition requires the trace of the gradient tensor to vanish: $\text{div} \, \mathbf{v} = d_1 + d_2 + d_3 = d_1(1 + \kappa_1 + \kappa_2) = 0$. For a non-trivial flow ($d_1 \neq 0$), we substitute the expressions for the similarity coefficients (11) for the physically realizable combination of signs $(-,-)$:

\begin{equation}

1 + \left(1 - \sqrt{\frac{a}{c}}\right) + \left(1 - \sqrt{\frac{a}{c}}\right)\left(1 - \sqrt{\frac{b}{a}}\right) = 3 - 2\sqrt{\frac{a}{c}} - \sqrt{\frac{b}{a}} + \sqrt{\frac{b}{c}} = 0

\end{equation}

Bringing to a common denominator $\sqrt{c}$ gives the exact algebraic relation (16). In this case, the nonlinear coefficient $\Gamma_{abc} \equiv 1 + \kappa_1^2 + \kappa_2^2 > 0$, which transforms the nonlinear term in the Riccati equation (14) into a damping source, eliminating the blow-up singularity under any initial conditions.

By Hunt's definition, the second invariant of the velocity gradient tensor for a divergence-free field is calculated as $Q = -\frac{1}{2} \text{tr}(\mathbf{G}^2)$. Expanding the trace of the square of the matrix $\mathbf{G} = \mathbf{S} + \boldsymbol{\Omega}$ via the components of the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ and the pure shear deformation tensor $\mathbf{S}_{\text{shear}}$, we obtain:

\begin{equation}

\text{tr}(\mathbf{G}^2) = (1 + \kappa_1^2 + \kappa_2^2) d_1^2 + 2|S_{\text{shear}}|^2 - \frac{1}{2}|\boldsymbol{\omega}|^2

\end{equation}

Multiplying by $-1/2$ and substituting the value of Γ_{abc} leads to the exact topological formula (17). Theorem 3 is proven.

\end{proof}

\begin{theorem}[On the Global Integral Closure of the Cauchy Problem in Free Space]

Let decaying Dirichlet boundary conditions be given at infinity of the space $\mathbb{R}^3$: $\lim_{|\mathbf{x}| \to \infty} \mathbf{v} = 0$. Then, under the conditions of the stabilizing resonance (16), the full pressure field is uniquely restored via the Newtonian potential:

\begin{equation}

p(\mathbf{x}, t) = p_\infty + \frac{\rho}{4\pi} \int_{\mathbb{R}^3} \frac{\sum_{i,j} \frac{\partial v_j}{\partial x_i'}(\mathbf{x}', t) \frac{\partial v_i}{\partial x_j'}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'

\end{equation}

Furthermore, the total kinetic energy is a monotonically decreasing function of time:

\begin{equation}

\frac{d}{dt} E_{\text{kin}}(t) = \frac{d}{dt} \left(\frac{\rho}{2} \int_{\mathbb{R}^3} |\mathbf{v}|^2 d\mathbf{x} \right) = -\rho \nu \int_{\mathbb{R}^3} |\nabla \mathbf{v}|^2 d\mathbf{x} \leq 0

\end{equation}

\end{theorem}

\begin{proof}

Applying the divergence operator to the original Navier--Stokes momentum equation under the condition $\text{div} \, \mathbf{v} = 0$ leads to the classical Poisson equation for pressure: $\Delta p = -\rho \sum_{i,j} \frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j}$. The convolution of the right-hand side with the Green's function for the free-space three-dimensional Laplace equation, assuming bounded pressure at infinity ($p \to p_\infty$), yields the identical integral representation (20).

To prove the energy identity (21), we multiply the equation of motion scalarly by $\mathbf{v}$ and integrate over the entire volume $\mathbb{R}^3$. Applying the Gauss--Ostrogradsky theorem (integration by parts), the boundary integrals vanish due to the decay of the Dirichlet fields at infinity. The convective term and the pressure gradient term vanish identically due to flow incompressibility:

\begin{equation}

\int_{\mathbb{R}^3} \mathbf{v} \cdot ((\mathbf{v} \cdot \nabla)\mathbf{v}) d\mathbf{x} = -\frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{v}|^2 (\nabla \cdot \mathbf{v}) d\mathbf{x} = 0

\end{equation}

\begin{equation}

\int_{\mathbb{R}^3} \mathbf{v} \cdot \nabla p \, d\mathbf{x} = -\int_{\mathbb{R}^3} p (\nabla \cdot \mathbf{v}) d\mathbf{x} = 0

\end{equation}

The viscous term transforms into a negatively definite Dirichlet integral, leading to a monotonic decrease in the kinetic energy norm (21). Theorem 4 is proven.

\end{proof}

\section{Conclusion}

In this work, a comprehensive analytical study of the modified Cauchy problem for the Navier--Stokes equations in the local neighborhood of the constraint functional's degeneracy point has been successfully completed. We have demonstrated that replacing the linear continuity equation with the nonlinear surrogate $Q = 0$ shifts the system into the class of differential-algebraic evolutionary systems prone to forming local singularities (gradient blow-up of the velocity profile in finite time).

Nevertheless, a resonant subspace of weight parameters $\sqrt{a} + \sqrt{b} = 2\sqrt{c}$ has been discovered, acting as a stabilizing attractor. Under this resonance, the nonlinearity loses its catastrophic character, the local law of mass conservation ($\text{div} \, \mathbf{v} = 0$) is fully restored, and the integral closure in the energy spaces $L^2(\mathbb{R}^3)$ proves the global-in-time correctness of the Cauchy problem for classes of viscous Stokes waves and dissipative Oseen--Taylor vortices. The obtained results can be utilized to construct stable numerical schemes for approximating hydrodynamic flows with nonlinear constraints.

\end{document}