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\documentclass[
aip,
pof, % Physics of Fluids
reprint, %
amsmath,amssymb
]{revtex4-2}
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\usepackage{dcolumn}
\usepackage{bm}
\usepackage[utf8]{inputenc}
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\begin{document}
\title{Mathematical Modeling of Hydrodynamic Flows in the Vicinity of Quasi-
Ruled Surfaces of Negative Curvature and the Topology of Turbulent Transition}
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\begin{abstract}

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This paper presents a novel hydro-geometric framework to investigate the localized kinematics and energy cascades of three-dimensional incompressible viscous flows. By introducing a spatial-phase parameterization of the strain rate tensor, we establish an exact localized non-linear invariant that maps fluid deformation directly onto the evolution of physical surfaces with negative Gaussian curvature. A rigorous qualitative analysis of the Navier--Stokes equations under this invariant reveals a structural topological singularity at the origin of the local coordinate system, which triggers a localized flow degeneration. We show that this degeneracy drives a fundamental metamorphosis of the governing pressure field operator, transforming it from a standard elliptic Poisson equation into a non-linear mixed hyperbolic-elliptic type where the local characteristic varieties act as sonic-like manifolds of the transition. For planar parallel shear flows, a structural degeneracy theorem is established, demonstrating that temporal changes under explicit spatial symmetries are strictly restricted to vorticity evolution (pure rotation) while entirely suppressing absolute velocity magnitude fluctuations. Furthermore, by framing the flow-field evolution within the context of structural mapping stability, we model the continuous physical transition from a stable laminar saddle configuration to a conical collapse. This catastrophic

geometric metamorphosis provides a strict deterministic model for the onset of localized turbulent vortices and wave jumps, where the rapid concentration of kinetic energy at the singular tip represents the precise macroscopic trigger for the laminar-to-turbulent transition.

`\end{abstract}`

`\keywords{Navier--Stokes equations, Differential forms, Killing vectors, Catastrophe theory, Conical collapse, Turbulent transition}`

`\maketitle`

`\section{Introduction and Model Formulation}`

Consider an unsteady flow of an incompressible viscous fluid in a three-dimensional Euclidean space. The classical system of Navier--Stokes equations includes the dynamic equations of motion and the linear law of conservation of mass (continuity equation):

`\begin{equation}`

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0$$

`\end{equation}`

The aim of this study is to parameterize the components of the strain rate tensor by introducing a spatial-phase argument  $\varphi$ :

`\begin{align}`

$$\frac{\partial v_1}{\partial x_1} = \varphi \cdot \cos \varphi \quad \text{\nonumber}$$

$$\frac{\partial v_2}{\partial x_2} = \varphi \cdot \cos \left( \varphi + \frac{2\pi}{3} \right) \quad \text{\nonumber}$$

$$\frac{\partial v_3}{\partial x_3} = \varphi \cdot \cos \left( \varphi - \frac{2\pi}{3} \right) \quad \text{\nonumber}$$

`\end{align}`

Representation (2) trivially satisfies relation (1) due to the trigonometric identity for shifted phases:

`\begin{equation}`

$$\varphi \cdot \left[ \cos \varphi + \cos \left( \varphi + \frac{2\pi}{3} \right) + \cos \left( \varphi - \frac{2\pi}{3} \right) \right] = 0$$

`\end{equation}`

Computing the difference of the deformation components along the second and third coordinate axes, we obtain:

`\begin{equation}`

$$\frac{\partial v_3}{\partial x_3} - \frac{\partial v_2}{\partial x_2} = \sqrt{3} \kappa \sin \varphi$$

Combining the equations, we eliminate the phase argument  $\varphi$  by transitioning to a quadratic form:

$$\left( \frac{\partial v_2}{\partial x_2} - \frac{\partial v_3}{\partial x_3} \right)^2 \cos^2 \varphi - 3 \left( \frac{\partial v_1}{\partial x_1} \right)^2 \sin^2 \varphi = 0$$

Normalizing expression (5) by the magnitude of the local acceleration vector of the medium  $\frac{\partial |v|}{\partial t}$ , we match it with the structure of the classical Euler formula for the normal curvature of surfaces. This isomorphism allows us to determine the principal curvatures of the associated virtual surface:

$$\begin{aligned} \kappa_1 &= \left( \frac{1}{\frac{\partial |v|}{\partial t}} \right) \cdot \left( \frac{\partial v_2}{\partial x_2} - \frac{\partial v_3}{\partial x_3} \right) \\ \kappa_2 &= \left( -\frac{3}{\frac{\partial |v|}{\partial t}} \right) \cdot \left( \frac{\partial v_1}{\partial x_1} \right) \end{aligned}$$

Integrating the local principal curvatures (6) into the second-order approximation equations (Monge form), we obtain a nonlinear equation of a quasi-ruled surface (a dynamic deformable hyperbolic paraboloid), which is a strict consequence of the flow kinematics.

### Analysis of the Cauchy Problem for the Modified System at Point M

Let us introduce a local Cartesian coordinate system  $(x, y, z)$ , aligning the origin  $(0,0,0)$  with the studied point of the medium  $M$  (where for simplicity of notation we set  $x_1=x, x_2=y, x_3=z$ ). To study local geometric instabilities and spatial-phase constraints, we introduce a theoretical mathematical ansatz. Instead of the classical equation, we require the fulfillment of a functional-geometric constraint throughout the neighborhood of point  $M$ :

$$y^2 \cdot \left( \frac{\partial v_2}{\partial y} - \frac{\partial v_3}{\partial z} \right)^2 - x^2 \cdot \left( \frac{\partial v_1}{\partial x} \right)^2 - 2z \cdot \frac{\partial |v|}{\partial t} = 0$$

### \subsection{Dimensional Homogeneity Analysis}

Compliance with the principle of dimensional homogeneity is critically important to substantiate the correctness of the model. The principal curvatures of a surface have the dimension  $\text{m}^{-1}$ . The spatial derivatives of the velocity field have the dimension  $\text{s}^{-1}$ , and the local acceleration has  $\text{m} \cdot \text{s}^{-2}$ . Dimensional analysis of the right-hand side of relation (6) yields:

$$\begin{aligned} & \left[ \left( \frac{1}{\frac{\partial |v|}{\partial t}} \right) \cdot \left( \frac{\partial v_i}{\partial x_j} \right)^2 \right] = \left( \frac{1}{\text{m}} \cdot \text{s}^{-2} \right) \cdot (\text{s}^{-1})^2 = \text{m}^{-1} \end{aligned}$$

Which formally confirms homogeneity. To strictly eliminate scale contradictions, a characteristic spatial scale of local flow inhomogeneity  $L_0$  is introduced, transforming the spatial basis into a dimensionless form.

### \subsection{Mathematical Features of Local Degeneracy: Tensor and Differential Form Formalism}

Let  $\mathcal{M}$  be a smooth 3-dimensional Riemannian manifold  $(\mathcal{M}, g)$  with a metric tensor  $g_{ij}$ . In a local chart around a fixed point  $M \in \mathcal{M}$  with coordinates  $x^i$ , the velocity field is represented as a smooth 1-form (co-velocity vector)  $v = v_i dx^i$ , and the acceleration field is given by the time-derivative  $\dot{v} = \frac{\partial v_i}{\partial t} dx^i$ . The classical velocity gradient tensor  $\nabla v$  can be decomposed into the symmetric strain rate tensor  $S = \frac{1}{2} \mathcal{L}_v g$  (where  $\mathcal{L}_v$  is the Lie derivative) and the antisymmetric vorticity tensor  $\Omega = \frac{1}{2} dv$ .

The coordination constraint (7), formulated under the algebraic parameterization of the components  $S_{ij}$ , can be intrinsically generalized on  $\mathcal{M}$  through the introduction of a symmetric tensor field of type  $(0,2)$ , denoted by  $\Phi = \Phi_{ij} dx^i \otimes dx^j$ , and a vector field  $\xi = \xi^i \frac{\partial}{\partial x^i}$  associated with the local positioning framework. Let  $\iota_\xi$  denote the interior product (contraction). The modified functional-geometric ansatz takes the following coordinate-free form:

$$\begin{aligned} & g(\iota_\xi \Phi, \iota_\xi \Phi) - 2 \cdot \iota_\xi \left( \star \left( \iota_\xi \dot{v} \wedge \star v \right) \right) = 0 \end{aligned}$$

where  $\star$  represents the Hodge star operator corresponding to the volume form  $\eta = \sqrt{|g|} dx^1 \wedge dx^2 \wedge dx^3$ , and  $\Phi$  encodes the non-linear interaction of the principal deformation components  $S_{ij}$ .

### \subsubsection{Topological Singularity and Underdetermination at the Origin}

Let  $\hat{x}^i = x^i$  define a local radial vector field vanishing precisely at the origin  $M$  ( $\hat{x}^i = 0$ ). At this singular point, the contraction operator  $\iota_{\hat{x}}$  degenerates identically:

$$\begin{aligned} \iota_{\hat{x}} \Phi|_M &= 0, \quad \iota_{\hat{x}} (\star \dot{v} \wedge \star v)|_M = 0 \end{aligned}$$

An consequence of this is that the non-linear geometric constraint equation collapses into the trivial identity  $0 = 0$ . On the tangent space  $T_M \mathcal{M}$ , the system of modified hydrodynamics equations loses one independent algebraic constraint, leading to a structural topological underdetermination. The velocity 1-form  $v$  and the pressure scalar field  $p$  become locally uncoupled through the boundary condition at the singularity  $M$ , transforming  $M$  into a focal point for geometric bifurcations.

### \subsubsection{Metamorphosis of the Pressure Operator Class}

In the standard incompressible Navier--Stokes formulation, applying the codifferential operator  $\delta = -\star d \star$  to the dynamic momentum equations yields a linear elliptic Hodge-Laplacian (Poisson) problem for the pressure field:  $\Delta p = \delta(v \cdot \nabla v)$ .

Under the modified framework, we introduce the exterior derivative  $d$  to the time-evolved constraint (9). Differentiating the non-linear metric constraint with respect to the temporal parameter  $t$  and invoking the dynamic equations via the interior product, the Hodge-Laplacian operator for  $p$  undergoes a fundamental structural metamorphosis. The application of  $\delta$  no longer generates a purely elliptic operator. Instead, the compatibility condition yields a first-order differential form for the scalar pressure field  $p$ , scaled by the contraction component  $z \cdot dx^3$ :

$$\iota_{\left(z \frac{\partial}{\partial z}\right)} dp + \mathcal{A}(v, S) = 0$$

where  $\mathcal{A}(v, S)$  represents a non-linear algebraic form containing higher-order contractions of the strain rate tensor.

The appearance of the directional derivative  $\iota_{\left(z \frac{\partial}{\partial z}\right)}$  introduces real characteristic varieties within the domain. This formally marks a topological transition of the governing system of partial differential equations from a classic parabolic-elliptic class to a degenerating system of mixed (hyperbolic-elliptic) type, where the surfaces of negative Gaussian curvature act as the sonic-like manifolds of the structural transition.

## \section{Exact Solutions under Killing-Form Invariance and Isometric Embeddings ( $v_3 = 0$ )}

To investigate the rigorous analytical properties of the modified hydrodynamic system, we restrict our domain to a specific geometric class where the velocity field possesses an explicit spatial symmetry. Let  $X = \frac{\partial}{\partial z}$  be a smooth vector field on the Riemannian manifold  $(\mathcal{M}, g)$ . We require  $X$  to be a Killing vector field, which means the Lie derivative of the metric tensor along  $X$  vanishes identically:

$$\begin{aligned} \begin{aligned} \mathcal{L}_X g &= 0 \end{aligned} \end{aligned}$$

In terms of differential forms, the metric dual 1-form (the Killing form)  $\xi = X_i dx^i = dz$  satisfies the fundamental Killing equation  $\nabla_i \xi_j + \nabla_j \xi_i = 0$ .

We impose the strict invariance of the velocity 1-form  $v$  and the pressure field  $p$  under the isometric flow generated by the Killing field  $X$ :

$$\begin{aligned} \begin{aligned} \mathcal{L}_X v &= 0, \quad \mathcal{L}_X p = 0 \end{aligned} \end{aligned}$$

Condition (13) implies that the third velocity component vanishes identically ( $v_3 = \iota_X v = 0$ ), and all remaining components  $v_1, v_2$  are strictly invariant with respect to the integral curves of  $X$  ( $\partial/\partial z = 0$ ). Under this Killing-form invariance, the non-linear coordination constraint (9) reduces to a functional relation restricted to the two-dimensional submanifold  $\mathcal{N} \subset \mathcal{M}$  (the  $Oxy$  plane):

$$\begin{aligned} \begin{aligned} y^2 \cdot \left( \iota_X \left( \frac{\partial}{\partial y} \right) dv \right)^2 - x^2 \cdot \left( \iota_X \left( \frac{\partial}{\partial x} \right) dv \right)^2 - 2 \cdot \iota_X \left( \star \dot{v} \wedge \star v \right) \right) &= 0 \end{aligned} \end{aligned}$$

**Steady-state Geodesic Flow Regime ( $\mathcal{L}_v g|_t = 0$ )**

Consider the case where the deformation kinetics are steady-state, meaning the absolute metric acceleration term vanishes,  $\frac{\partial |v|}{\partial t} = 0$ . The unsteady contraction block in (14) is eliminated, mapping the fluid kinematics onto a purely stationary geometric problem on  $\mathcal{N}$ :

$$\begin{aligned} \begin{aligned} \left| y \cdot \iota_X \left( \frac{\partial}{\partial y} \right) dv \right| &= \left| x \cdot \iota_X \left( \frac{\partial}{\partial x} \right) dv \right| \end{aligned} \end{aligned}$$

Applying the method of separation of variables to the exterior differential forms in (15), we obtain a self-similar class of exact solutions. The velocity 1-form  $v$  can be integrated on the entire plane  $\mathcal{N} \setminus \{x=0, y=0\}$  into a logarithmic representation:

$$\begin{aligned} \begin{aligned} v_1(x, y) &= A_1 \ln|x| + \psi(y) \\ v_2(x, y) &= A_2 \ln|y| + \omega(x), \quad \text{subject to } |A_1| \end{aligned} \end{aligned}$$

$$= |A_{-2}|$$

\end{align}

where  $\psi(y)$  and  $\omega(x)$  are smooth arbitrary functions governed by the boundary conditions on the submanifolds of  $\mathcal{N}$ .

\subsection{Theorem of Structural Degeneracy under Killing Symmetry}

\textbf{Theorem 1.} \textit{Let  $v$  be an unsteady incompressible fluid flow on  $(\mathcal{M}, g)$  satisfying the modified non-linear constraint (9). If  $v$  is invariant under the action of the Killing vector field  $X = \frac{\partial}{\partial z}$ , then the kinetic energy density of any local fluid volume element is invariant in time, i.e.,  $\frac{\partial |v|}{\partial t} = 0$ .}

\textit{Proof.} Suppose  $\frac{\partial |v|}{\partial t} \neq 0$ . Due to the explicit linear dependence of the uncoupled contraction term on the coordinate function  $z$  (induced by the projection  $\iota_X$ ), equation (14) can be split into independent structural components along the fibers of the tangent bundle. Since the velocity components  $v_1, v_2$  belong to the kernel of  $\mathcal{L}_X$  by definition (13), they cannot depend on  $z$ . Therefore, the only mathematically consistent configuration that satisfies equation (14) globally across the entire neighborhood of the point  $M$  is the simultaneous and identical zeroing of both the spatial differential forms and the temporal evolution term:

\begin{equation}

$$\iota_X \left( \star \left( \star \dot{v} \wedge \star v \right) \right) = 0 \quad \text{and} \quad y^2 \cdot \left( \iota_{\frac{\partial}{\partial y}} dv \right)^2 - x^2 \cdot \left( \iota_{\frac{\partial}{\partial x}} dv \right)^2 = 0$$

\end{equation}

This immediately yields  $\frac{\partial |v|}{\partial t} = 0$ , which contradicts the initial assumption. Thus, any temporal evolution under Killing symmetry is strictly restricted to non-trivial changes in the vorticity 2-form  $dv$  (pure rotation), completely banning any dynamic fluctuations in the absolute magnitude of the local velocity field.  $\blacksquare$

\section{Topologies of Turbulent Transition via René Thom's Catastrophe Theory}

To describe the non-linear evolution of the flow geometry over time, we abandon purely empirical kinematic parameters and reformulate the structural deformation of the medium within the framework of René Thom's catastrophe theory and Whitney's theory of differentiable mappings. Let us consider the dynamic flow restructuring as the evolution of a smooth family of target manifolds parameterized by a time-dependent control vector.

We define a smooth potential-like function  $\mathcal{F}: \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{R}$ , whose zero-sets describe the local evolution of the fluid volume boundaries in the Monge form:

\begin{equation}

$$\mathcal{F}(x, y, z; \epsilon) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 2z - \epsilon z^2 = 0$$

\end{equation}

where  $\varepsilon(t) \in \mathbb{R}$  acts as a dynamic control parameter (deformation parameter) governed by the non-linear constraint (9).

According to Thom's classification theorem, the smooth family of surfaces (18) undergoing a transition through  $\varepsilon = 0$  structurally mimics the canonical cusp catastrophe (Whitney's fold and cusp singularities) on the 3D differential manifold. The sign of the control parameter  $\varepsilon(t)$  explicitly dictates the global topological state of the continuum:

```
\begin{itemize}
\item \textbf{Laminar Shear Sheet ( $\varepsilon = 0$ ):} The manifold  $\mathcal{F}^{-1}(0)$  is locally diffeomorphic to a classical hyperbolic paraboloid (smooth saddle geometry), representing stable laminar flow streamlines.
\item \textbf{Supercritical One-Sheeted Hyperboloid ( $\varepsilon > 0$ ):} The configuration space undergoes a topological modification into a connected single-sheeted hyperboloid, physically corresponding to the formation of stable, continuous fluid filaments.
\item \textbf{Subcritical Two-Sheeted Hyperboloid ( $\varepsilon < 0$ ):} The manifold breaks symmetry into a disconnected two-sheeted hyperboloid, marking the structural tearing of the localized continuum and the onset of localized structural defects.
\end{itemize}
```

**Cusp Singularity and the Differential Geometry of Geometric Evolution**

Let us analyze the critical points of the mapping  $\mathcal{F}$ . The singular locus (the catastrophe manifold)  $\Sigma \subset \mathcal{M}$  is defined by the vanishing of the differential with respect to the internal coordinate function  $z$ :

```
\begin{equation}
\frac{\partial \mathcal{F}}{\partial z} = -2 - 2\varepsilon(t) z = 0 \implies z_c = -\frac{1}{\varepsilon(t)}
\end{equation}
```

The bifurcation set (the fold line) in the control space occurs when the second derivative also vanishes,  $\frac{\partial^2 \mathcal{F}}{\partial z^2} = -2\varepsilon(t) = 0$ , triggering a structural jump directly at  $\varepsilon \rightarrow 0$ .

Suppose that during a finite temporal increment  $\Delta t$ , the system transitions from the initial state (18) to a perturbed target manifold governed by Whitney's structural stability condition:

```
\begin{equation}
\frac{x^2}{a^2} + \varepsilon \cdot \frac{z^2}{c^2} - \frac{y^2}{b^2} = \varepsilon
\end{equation}
```

By evaluating the total derivative of the parameter  $\varepsilon(t)$  along the structural transition trajectories between the unperturbed state (18) and the perturbed state (20), we pass to the standard differential form for the velocity

of geometric evolution  $\partial \varepsilon / \partial t$ :

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{\Delta t} \left[ \frac{c^2}{a^2} \left( \frac{y^2}{a^2} - \frac{x^2}{b^2} \right) - \varepsilon \right]$$

Integrating (21) yields the exact implicit equation for the moving boundary  $F(x, y, z, t) = 0$ :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 2z - z^2 \left[ \frac{\varepsilon_0}{\Delta t} + \frac{1}{\Delta t} \left( \frac{c^2}{a^2} \left( \frac{y^2}{a^2} - \frac{x^2}{b^2} \right) - \varepsilon_0 \right) \right] = 0$$

### Conical Collapse and the Metamorphosis into Turbulent Vortices

We substitute the geometric constants with the principal curvatures of the associated flow field, setting  $b^2 = \frac{c}{|\kappa_1|}$  and  $a^2 = \frac{c}{|\kappa_2|}$ . At the critical temporal threshold  $t = \Delta t$ , the dynamic system (22) reaches the bifurcation boundary:

$$F(x, y, z; t + \Delta t) = |\kappa_2| x^2 + \frac{\varepsilon}{c} z^2 - |\kappa_1| y^2 - \frac{c}{\varepsilon} = 0$$

As  $\varepsilon \rightarrow 0$ , the term  $\frac{c}{\varepsilon}$  forms a co-dimension 2 singularity. The smooth manifold experiences a catastrophic topological metamorphosis---a conical collapse (conical singularity):

$$|\kappa_2| x^2 - |\kappa_1| y^2 = 0$$

The cross-section of the flow degenerates identically into a pair of intersecting planes (an asymptotic cone), forming sharp geometric edges where the classical differentiability of the velocity field is lost.

**Physical and Topological Interpretation:** The transition from a smooth, non-compact saddle surface ( $\varepsilon = 0$ ) to a conically collapsed state (24) mathematically models the macroscopic mechanism of the laminar-to-turbulent transition. Localized turbulent vortex tubes originate precisely at these singular conical vertices. At the moment of the catastrophe, the kinetic energy concentrates at the singular tip  $z_c$ , generating a localized wave jump (shock-like structure) that mimics the formation of a turbulent eddy within a smooth velocity field.

### Conclusion

An integrated nonlinear model linking the Navier--Stokes equations with the dynamic evolution of quasi-ruled surfaces is proposed. It is shown that introducing the geometric invariants and the time parameter  $\varepsilon(t)$  through

Thom's catastrophe framework allows a mathematically rigorous description of the restructuring of a laminar hydrodynamic field into a turbulent one through spatial bifurcations of the Monge form. The resulting wave jumps and conic sections open new geometric methods for analyzing the stability of viscous fluid flows.

\section\*{Data Availability Statement}

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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